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THE CONTROL OF LARGE DEVIATIONS IN OSCILLATORY SYSTEMS WITH SMALL RANDOM PERTURBATIONS†

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The problem of controlling a system, reduced to the standard form with a small parameter ε, is considered. The control is supposed to be weak, while the perturbation is a broadband process proportional to the small parameter. If the unperturbed system has an asymptotically stable stationary point, then most of perturbed trajectories are attracted to this point, and the average time the system resides in a bounded region approaches infinity for small ε. Small random perturbations can lead to rare large deviations from the unperturbed trajectory and to escape from the region within a comparatively short time. Rare events scarcely affect the average residence time, and an exponential criterion of a special form is introduced to solve the optimal control problem taking account of large deviations. The criterion depends on the parameter e and converts into the maximum residence time functional for deterministic systems. An asymptotic solution of the dynamic programming equation is constructed. It is shown that the main term of the asymptotic solution depends on the perturbation intensity, although in the original equations this quantity is small. The control problem for a pendulum is considered as an example. A control is constructed that holds the system in a specified region of oscillatory motions within the maximum time interval.

Criteria of the Mayer or Boltza type, which are normally used in control problems for mechanical systems, are ineffective for control which takes large deviations into account. When these criteria are used, it is assumed that the system dynamics is similar to the unperturbed dynamics, and small random perturbations lead to small fluctuations in the trajectories. In this case, the problem is reduced to controlling the deterministic part of the system, and allowance for small perturbations leads to small corrections in the main control circuit. Such an approach is acceptable if escape of the trajectory from the admissible region does not lead to irreversible breakdowns in system operation. If, however, escape from the region is regarded as a dangerous event, then the control must be sensitive to small perturbations. Examples of this are guidance, tracking and synchronization problems, certain problems of financial risk, etc. [1–3].

To construct a control that is sensitive to small perturbations, exponential criteria depending on a small parameter have been proposed [1–3]. For a deterministic system, these criteria are converted into Mayer or Boltza functionals. The effectiveness of the exponential counterpart of the minimum time criterion in the problem of controlling large deviations in systems with time-independent drift and diffusion coefficients has been demonstrated [3].

In the present paper, a similar approach is developed for oscillatory systems which permit of large deviations. The dynamic programming equation for a system perturbed by white noise is constructed. The solution is sought by the averaging method. An extension of this approach to systems with rotating phase and systems with broadband perturbations other than white noise is proposed.

1. FUNDAMENTAL EQUATIONS AND FUNCTIONALS OF THE PROBLEM

The equations of motion are reduced to the standard form with a small parameter ε

$$x' = \varepsilon F(t,x,u) + \varepsilon\sigma(t,x)w'(t), \quad x(t) = x \in G; \quad x(0) = x^*; \quad u \in U \tag{1.1}$$

Here G is the open region in R_n with boundary Γ, U is a compactum in R_m, $w(t)$ is a standard Wiener process in R_l, and the prime denotes a derivative with respect to fast time t. The drift coefficient $F(t, x, u)$ is assumed to be sufficiently smooth with respect to all variables and to be periodic or almost

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periodic and uniformly bounded for all $t \in I_T = (-\infty, \infty)$ in the domain D: $\{I_T \times \bar{G} \times U\}$. The diffusion matrix

$$a(t,x) = \sigma(t,x)\sigma^T(t,x)$$

satisfies similar conditions in the domain Q: $\{I_T \times \bar{G}\}$. In addition, it is assumed that the matrix $a(t,x)$ is non-degenerate and positive definite in Q.

To distinguish the main time-scale, we will introduce the slow variable $\tau = \varepsilon t$ and rewrite (1.1) in the form

$$\begin{aligned} \dot{x}^\varepsilon &= F^\varepsilon(\tau, x^\varepsilon, u) + \varepsilon^{1/2}\sigma^\varepsilon(\tau, x^\varepsilon)\dot{w}(\tau), \quad x^\varepsilon(\tau) = x, \quad x^\varepsilon(0) = x^* \\ F^\varepsilon(\tau, x, u) &= F(\tau/\varepsilon, x, u), \quad \sigma^\varepsilon(\tau, x) = \sigma(\tau/\varepsilon, x) \end{aligned} \tag{1.2}$$

where the dot denotes the derivative with respect to slow time τ. The diffusion matrix

$$a^\varepsilon(\tau, x) = \sigma^\varepsilon(\tau, x)(\sigma^\varepsilon(\tau, x))^T$$

As a rule, the criterion chosen to characterize the system reliability is the average time the system resides in the specified region

$$\Phi^\varepsilon(u) = \mathrm{M}\tau_G \tag{1.3}$$

where M is the mathematical expectation and τ_G is the time of the first exit from the region

$$\tau_G = \inf\{\tau : x^\varepsilon(\tau) \neq G / x^\varepsilon(0) = x^*\} \tag{1.4}$$

Suppose $F^\varepsilon(\tau, x, 0) = f^\varepsilon(\tau, x)$. We will assume that the generating uncontrolled system

$$\dot{x}^\varepsilon = f^\varepsilon(\tau, x^\varepsilon) \tag{1.5}$$

satisfies the conditions of applicability of the averaging method [4, 5] and has an asymptotically stable equilibrium position defined as a stationary solution of the averaged system

$$\dot{x}^0 = f^0(x^0), \quad f^0(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,x)dx \tag{1.6}$$

If all the admissible initial points lie in the domain of attraction of this equilibrium position, then, with a probability close to unity, the majority of the trajectories of perturbed system (1.2) falls into a small neighbourhood of it and stays there for an arbitrary long time, provided ε is sufficiently small. The probability of the system residence in the region G lying in the domain of attraction of the asymptotically stable position and the average first escape time are estimated exponentially [6, 7[

$$\begin{aligned} &\mathrm{P}\{\tau_G \leqslant T\} \sim \exp(-C/\varepsilon), \quad C > 0, \quad T \sim 1 \\ &\mathrm{M}\tau_G \sim \exp(-C_1/\varepsilon), \quad C_1 > 0 \end{aligned} \tag{1.7}$$

It follows from relations (1.7) that the main contribution to criterion (1.3) is made by trajectories with exponentially large residence time. This implies that the control Based on the maximum criterion (1.3) takes no account of dangerous escapes at relatively short time. The problem is to construct a control sensitive to large deviations at relatively short time intervals and, accordingly, to external exitation. To solve this problem, it is proposed to introduce the following exponential criterion [1, 2]

$$\Phi^\varepsilon(u) = \mathrm{M}\exp(-\theta\tau_G/\varepsilon) \tag{1.8}$$

or, in a more general case,

$$\Phi^\varepsilon(u) = \mathrm{M}\exp\left[-\varepsilon^{-1}\int_0^{\tau_G} L(x,u)d\tau\right] \tag{1.9}$$

where $\theta > 0$ is a weight coefficient and $L(x, u)$ is a sufficiently smooth function. Minimization of functional (1.8) corresponds to maximizing criterion (1.3).

It is obvious that the main contribution to criteria (1.8) and (1.9) is made by trajectories with a comparatively small residence time in the region G. This means that the exponential criteria are "shifted" towards more dangerous events, associated with escape from the region within a short time.

Problems (1.2), (1.8) and (1.2), (1.9) were considered in detail in [2] for the case of uniform convergence of the coefficients $F^\varepsilon \to F^0$ and $\sigma^\varepsilon \to \sigma^0$ as $\varepsilon \to 0$. The conditions for a solution of the optimal control problem to exist were formulated, a dynamic programming equation was constructed and the passage to the limit as $\varepsilon \to 0$ was proved. Weaker conditions were proposed [3] for a system with coefficients independent of τ and ε. In the present paper, the results are extended to systems with averaging, i.e. with the integral convergence $F^\varepsilon \to F^0$ and $\sigma^\varepsilon \to \sigma^0$ [4]. The problem of the existence of a solution is not considered. It is assumed that the smoothness conditions on the coefficients ensure the existence of a solution of the dynamic programming equation and the correctness of the necessary transformations. Our purpose is to construct an asymptotic solution (as $\varepsilon \to 0$) of problem (1.2) with criterion (1.8) or (1.9).

2. THE DYNAMIC PROGRAMMING EQUATIONS

We will construct the dynamic programming equations for problem (1.2), (1.8). We will define the Bellman function

$$W^\varepsilon(\tau, x) = \min_{u\in U} \mathrm{M}_{\tau,x} \exp(-\theta\tau_G/\varepsilon) \tag{2.1}$$

where $M_{\tau,x}$ is the conventional mathematical expectation with respect to the state $x^\varepsilon(\tau) = x$. Function (2.1) will be sought as the solution of the dynamic programming equation [1, 2]

$$W_\tau^\varepsilon + \frac{\varepsilon}{2}\mathrm{Tr}[a^\varepsilon(\tau,x)W_{xx}^\varepsilon] + h^\varepsilon(\tau,x,W_x^\varepsilon) - \frac{\theta}{\varepsilon}W^\varepsilon = 0, \quad x \in G \tag{2.2}$$
$$W^\varepsilon(\tau, x) = 1, \quad x \in \Gamma$$

Here the subscripts denote partial derivatives with respect to the corresponding variables and

$$h^\varepsilon(\tau,x,p) = \min_{u\in U}(F^\varepsilon(\tau,x,u),p) \tag{2.3}$$

where (F^ε, p) is the scalar product of the corresponding vectors.

As $\varepsilon \to 0$, Eq. (2.2) is singular, and its solution weakly depends on random perturbations and decreases exponentially with respect to the parameter $1/\varepsilon$ in the interval $\tau_G \sim 1$. To regularize the problem, a new variable $V^\varepsilon(\tau, x)$ is introduced according to the formula [1, 2]

$$W^\varepsilon(\tau, x) = \exp[-V^\varepsilon(\tau, x)/\varepsilon] \tag{2.4}$$

or

$$V^\varepsilon(\tau,x) = -\varepsilon \ln W^\varepsilon(\tau,x) = \max_{u\in U}\{-\varepsilon\ln[\mathrm{M}_{\tau,x}\exp(-\theta\tau_G/\varepsilon)]\} \tag{2.5}$$

Using replacement of the variable (2.4), (2.5) we transform relations (2.2) and (2.3) into the equation for the function $V^\varepsilon(\tau, x)$ with the corresponding boundary condition

$$V_\tau^\varepsilon + \frac{\varepsilon}{2}\mathrm{Tr}[a^\varepsilon(\tau,x)V_{xx}^\varepsilon] + H^\varepsilon(\tau,x,V_x^\varepsilon) - \frac{1}{2}(a^\varepsilon(\tau,x)V_x^\varepsilon, V_x^\varepsilon) = 0, \quad x \in G \tag{2.6}$$
$$V^\varepsilon(\tau, x) = 0, \quad x \in \Gamma$$

where

$$H^\varepsilon(\tau,x,p) = \max_{u\in U}(F^\varepsilon(\tau,x,u),p) + \theta \tag{2.7}$$

The solution of problem (2.6), (2.7) has the form

$$u^\varepsilon(\tau,x)=U(\tau/\varepsilon,x,V_x^\varepsilon(\tau,x)) \tag{2.8}$$

As follows from (2.6), as $\varepsilon\to 0$, the function $V^\varepsilon(\tau,x)$ depends on the diffusion matrix a^ε in the leading order, although in the equation of motion this quantity occurs with the coefficient ε. Thus, criterion (2.5) is sensitive to external exitation. If the uniformly bounded solution of (2.6) exists as $\varepsilon\to 0$, then the main contribution to (2.5) is made by large deviations with the residence time $\tau_G\sim 1$, i.e. the control follows escape from the region at finite rather than at exponentially large time intervals.

It is obvious that, in deterministic systems ($a^\varepsilon=0$), function (2.6) is identical with the Bellman function

$$V(\tau,x)=\max_{u\in U}(\theta\tau_G)$$

In stochastic systems, the solution of problem (2.5)–(2.7) corresponds to maximization of the functional

$$\Phi^\varepsilon(u)=-\varepsilon\ln[\mathrm{M}\exp(-\theta\tau_G/\varepsilon)] \tag{2.9}$$

This expression can be regarded as a regularized analogue (as $\varepsilon\to 0$) of functional (1.3). Maximization of criterion (2.9) instead of the standard criterion (1.3) enables as to construct a control law for the system that is consistent with the physical meaning of the problem.

When the coefficients are sufficiently smooth a classical solution of Eq. (2.6) exists that determines the solution of optimal control problem (2.5) on trajectories of system (1.2) [2, 3]. On taking the limit as $\varepsilon\to 0$, the procedure of the perturbation method is used in the dynamic programming equations [8, 9]. In the limit as $\varepsilon\to 0$, we obtain the averaged equation

$$H^0(x,V_x^0)-\frac{1}{2}(a^0(x)V_x^0,V_x^0)=0,\quad x\in G \tag{2.10}$$
$$V^0(x)=0,\quad x\in\Gamma$$

where, for the functions H^ε and a^ε, denoted by $G^\varepsilon(\tau,x,p)$, the limit is defined [5] as

$$G^0(x,p)=\lim_{\varepsilon\to 0}G^\varepsilon(\tau,x,p)=\lim_{\varepsilon\to 0}G(\tau/\varepsilon,x,p)=\lim_{T\to\infty}\frac{1}{T}\int_0^T G(t,x,p)dt \tag{2.11}$$

The existence of a limit $V^0(x)$ as the viscosity solution of the equation was proved earlier [2, 3, 8]. The convergence

$$V^\varepsilon(\tau,x)\to V^0(x),\quad \varepsilon\to 0,\quad \tau,x\in Q_G:\{[0,\tau_G]\times\overline{G}\} \tag{2.12}$$

is uniform on each compact $\mu m Q_G$. The control

$$u^0(t,x)=U(t,x,V_x^0(x)) \tag{2.13}$$

is regarded as quasioptimal. The quasioptimality of (2.13) can be proved using the standard procedure [10].

The solution of problem (1.2), (1.9) is constructed using similar procedures. The function

$$V^\varepsilon(\tau,x)=\max_{u\in U}\left\{-\varepsilon\ln\left[\mathrm{M}_{\tau,x}\exp\left(-\varepsilon^{-1}\int_0^{\tau_G}L(x,u)d\tau\right)\right]\right\} \tag{2.14}$$

is defined as the solution of Eq. (2.6), in which

$$H^\varepsilon(\tau,x,p)=\max_{u\in U}[(F^\varepsilon(\tau,x,u),p)+L(x,u)] \tag{2.15}$$

The Bellman function (2.14) is approximated by the solution of the averaged equation (2.10), and $H^0(x,p)$ is determined by averaging expression (2.15).

3. SOME GENERALIZATIONS

1. Suppose the equations of motion are reduced to the standard form of equations with a rotating phase

$$
\begin{aligned}
&x' = \varepsilon[F_1(\Psi,x,u)+\sigma_{11}(\Psi,x)w_1'(t)+\sigma_{12}(\Psi,x)w_2'(t)]\\
&\Psi' = \omega(x)+\varepsilon[F_2(\Psi,x,u)+\sigma_{21}(\Psi,x)w_1'(t)+\sigma_{22}(\Psi,x)w_2'(t)]\\
&x\in\overline{G}\in R_n,\quad \Psi\in R_1
\end{aligned}
\tag{3.1}
$$

where $w_1(t)$ and $w_2(t)$ are independent Wiener processes of relevant dimensionality. The right-hand sides of Eqs (3.1) are assumed to be sufficiently smooth with respect to all the variables and 2π-periodic with respect to the phase Ψ in the region considered; the frequency $\omega(x) \geqslant \omega_0 > 0$ when $x = \overline{G}$.

Suppose the problem consists of minimizing functional (1.9) on trajectories of system (3.1). Without introducing the slow variable τ, we will rewrite (1.9) in the form

$$
\Phi^\varepsilon(u) = \mathrm{M}\exp\left[-\int_0^{T_G} L(x,u)dt\right]
\tag{3.2}
$$

where $T_G = \inf\{t: x(t,\varepsilon) \notin G\}$, and define the Bellman function

$$
V^\varepsilon(\Psi,x) = \max_{u\in U}\left\{-\varepsilon\ln\left[\mathrm{M}_{\Psi,x}\exp\left(-\int_0^{T_G} L(x,u)dt\right)\right]\right\}
\tag{3.3}
$$

as the solution of the equation

$$
\begin{aligned}
&\omega(x)V_\Psi^\varepsilon+\varepsilon H(\Psi,x,V_x^\varepsilon,V_\Psi^\varepsilon)-\frac{\varepsilon}{2}[(a_{11}(\Psi,x)V_x^\varepsilon,V_x^\varepsilon)+[a_{12}(\Psi,x)V_x^\varepsilon,V_\Psi^\varepsilon)+\\
&+(a_{21}(\Psi,x)V_\Psi^\varepsilon,V_x^\varepsilon)+(a_{22}(\Psi,x)V_\Psi^\varepsilon,V_\Psi^\varepsilon)]+\varepsilon^2\ldots=0,\quad x\in G\\
&V^\varepsilon(\Psi,x)=0,\quad x\in\Gamma
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
&a_{ik}=\sigma_{ik}(\sigma_{ik})^T\\
&H(\Psi,x,p,r)=\max_{u\in U}[(F_1(\Psi,x,u),p)+(F_2(\Psi,x,u),r)+L(x,u)]\\
&u=U(\Psi,x,p,r)
\end{aligned}
\tag{3.5}
$$

As in Section 2, taking the limit as $\varepsilon\to 0$ corresponds to averaging with respect to the fast phase Ψ and to neglecting higher-order terms in ε. As a result, we obtain an equation analogous to (2.10) with coefficients independent of Ψ

$$
a^0(x)=\frac{1}{2\pi}\int_0^{2\pi} a_{11}(\Psi,x)d\Psi,\quad H^0(x,p)=\frac{1}{2\pi}\int_0^{2\pi} H(\Psi,x,p)d\Psi
\tag{3.6}
$$

Following the approach described in [8, 11], it is possible to prove the uniform convergence

$$
V^\varepsilon(\Psi,x)\to V^0(x)
$$

as $\varepsilon\to 0, x\in G$.

The quasioptimal control u^0 is written in the form

$$
u^0=U(\Psi,x,V_x^0(x),0)
\tag{3.7}
$$

The function $U(\Psi,x,p,r)$ can be found from the final relation of (3.5).

2. Consider a system with perturbations different from white noise

$$
x'=\varepsilon F(t,x,u)+\varepsilon\gamma(t,x)\xi(t)
\tag{3.8}
$$

where $\xi(t)$ is a stationary random process with a zero mean, satisfying the mixing conditions valid in particular, for stationary Gaussian processes [6, 10]. The coefficients F and γ are assumed to be sufficiently smooth in the domain D: $\{I_T \times \bar{G} \times U\}$.

Using the procedure of diffusion approximation [12], we approximate (3.8) with the partially averaged system

$$\dot{x}^\varepsilon = F^\varepsilon(\tau, x^\varepsilon, u) + \varepsilon B(x^\varepsilon) + \varepsilon^{1/2}\sigma(x^\varepsilon)\dot{w}(\tau) \tag{3.9}$$

The additional term $\varepsilon B(x)$ corresponds to the Stratonovich correction [6] and can be calculated in the explicit form [12]. In the present problem, the form of $B(x)$ is insubstantial, since the terms $O(\varepsilon)$ in the drift coefficient do not occur in the limiting equation (2.10).

The averaged diffusion matrix $a(x) = \sigma(x)\sigma^T(x)$ is calculated by means of the formula

$$a(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \gamma(t,x)K(s)\gamma^T(t+s,x)\,ds\,dt \tag{3.10}$$

where $K(s)$ is the correlation matrix of the process $\xi(t)$. Similar transformations may be carried out for system (3.1) with a rotating phase.

4. EXAMPLE. CONTROL OF A QUASICONSERVATIVE SYSTEM, PREVENTING FROM EXIT THROUGH A POTENTIAL BARRIER

Problems of controlling large deviations are of special interest if the unperturbed system is asymptotically stable. At the same time, such a formulation of the problem also has meaning for non-asymptotically stable systems. In this case, the control is "sensitive" to external exitation and increases the average residence time in the region.

We, will examine a quasiconservative system whose dynamics are described by the equations

$$q' = E_v, \quad v' = -E_q + \varepsilon u + \varepsilon\sigma w'(t) \tag{4.1}$$

where the function

$$E = v^2/2 + Q(q) \tag{4.2}$$

determines the energy integral of the unperturbed conservative system. It is assumed that the potential $Q(q)$ allows of a stable equilibrium position q_s

$$dQ/dq = 0, \quad d^2Q/dq^2 > 0 \quad \text{when} \quad q = q_s$$

and an unstable equilibrium position q_u

$$dQ/dq = 0, \quad d^2Q/dq^2 < 0 \quad \text{when} \quad q = q_u$$

Then, the motion in the region $Qq_s) \leqslant Q(q) < Q(q_u)$ corresponds to oscillations within a potential well; a transition through the potential barrier corresponds to the intersection of the separatrix separating regions with different types of motion and is regarded as a dangerous event. The aim of the control is to prevent random jumps through the potential barrier.

Reduce (4.1) to the standard form (3.1). We introduce the new variables e, and Ψ according to the formulae [5]

$$e = \frac{v^2(e,q)}{2} + Q(q), \quad \frac{\partial\Psi}{\partial q} = \frac{\omega(e)}{v(e,q)}, \quad \omega(e) = \frac{2\pi}{T(e)}$$
$$v(e,q) = \pm[2(e - Q(q(e,\Psi))]^{1/2}, \quad T(e) = \oint \frac{dq}{v(e,q)} \tag{4.3}$$

(integration is carried out along the contour e = const). Formally, the boundary of the admissible region is defined by the separatrix equation $E = e^* = Q(q_u)$; the actual admissible region is determined by the condition G: $\{e \leqslant e_\Gamma < e^*\}$.

Replacement (4.3) transforms Eqs (4.1) to the form [5]

$$e' = \varepsilon u v + \varepsilon^2\sigma^2/2 + \varepsilon\sigma v w'(t)$$
$$\Psi' = \omega(e) + \varepsilon g(e, \Psi, u) + \varepsilon\delta(e, \Psi) w'(t) \tag{4.4}$$

From (4.4) it follows that, for an uncontrolled system ($u = 0$), the average residence time in the finite region $M/\tau_G \sim 1/\varepsilon^2$. The introduction of a control makes the system dissipative, with an exponentially large residence time in region G.

We will examine two types of constraint and the functionals of the problem.

1. We will construct a control that maximizes the functional

$$\Phi^\varepsilon(u) = -\varepsilon \ln\left\{\mathrm{M}\exp\left[-\int_0^{T_G} \omega(e)dt\right]\right\} \tag{4.5}$$

on trajectories of (4.4), subject to the condition $|u| \le U$. Here, $t = T_G$ is the time when the trajectory first reaches the boundary of the region $e = e_\Gamma$. As in deterministic systems [10, 13], functional (4.5) corresponds not to the time but to the phase of escape from the domain G.

Equation (3.4) for problem (4.4), (4.5) can be written in the form

$$\omega(e)V_\Psi^\varepsilon + \varepsilon H(\Psi, e, v, V_e^\varepsilon) - \frac{\varepsilon}{2}(\sigma v V_e^\varepsilon)^2 + \varepsilon^2 \ldots = 0, \quad e \in G$$
$$V^\varepsilon(\Psi, e) = 0, \quad e = e_\Gamma \tag{4.6}$$

where

$$v = v(e, q(e, \Psi)), \quad H(\Psi, e, v, V_e^\varepsilon) = \max_{|u| \le U}(uv V_e^\varepsilon) + \omega(e) \tag{4.7}$$

The second equality of (4.7) indicates that

$$u^\varepsilon = U \operatorname{sgn}(v V_e^\varepsilon), \qquad H = U\,|v V_e^\varepsilon| + \omega(e) \tag{4.8}$$

Substituting the function H from (4.8) into (4.6) and averaging in the fast phase Ψ, we obtain the reduced dynamic programming equation (2.10) in the form

$$U\,|V_e^0|\,S(e) - \tfrac{1}{2}(\sigma V_e^0)^2 I(e) = -1, \quad e \in G$$
$$V^0(e) = 0, \quad e = e_\Gamma \tag{4.9}$$

where $I(e) > 0$ is the action integral [5]

$$I(e) = \frac{1}{2\pi}\int_0^{2\pi} \frac{v^2(e, q(e, \Psi))}{\omega(e)} d\Psi = \frac{1}{2\pi}\oint v(e, q)dq \tag{4.10}$$

and $S(e) > 0$ is the normalized length of the integration contour

$$S(e) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|v(e, q(e, \Psi))|}{\omega(e)} d\Psi = \frac{1}{2\pi}\oint \operatorname{sgn} v\, dq \tag{4.11}$$

It is obvious that the solution of Eq. (4.9) exists for any $U > 0$ and $\sigma > 0$; $V_e^0 < 0$ corresponds to the solution of the control problem. If $V_e^0 < 0$, from relations (3.7) and (4.8) when $V_e^0 < 0$ we obtain the quasioptimal control in the form

$$u^0 = -U \operatorname{sgn} v \tag{4.12}$$

2. Consider another type of functional

$$\Phi^\varepsilon(u) = -\varepsilon \ln\left\{\mathrm{M}\exp\left[-\int_0^{T_G} [\omega(e) - u^2/2r^2]dt\right]\right\} \tag{4.13}$$

Equation (3.4) for problem (4.4), (4.13) can be written in the form of (4.6), where

$$H(\Psi, e, v, V_e^\varepsilon) = \max_u (uvV_e^\varepsilon - u^2/2r^2) + \omega(e) \tag{4.14}$$

From (4.14) it follows that

$$u^\varepsilon = rvV_e^\varepsilon, \quad H = (rvV_e^\varepsilon)^2/2 + \omega(e) \tag{4.15}$$

Substituting (4.15) into (4.6) and averaging in the fast phase Ψ, we obtain the reduced dynamic programming equation (2.10) in the form

$$\tfrac{1}{2}(r-\sigma^2)(V_e^0)^2 I(e) = -1, \quad e \in G \tag{4.16}$$
$$V^0(e) = 0, \quad e = e_\Gamma$$

where $I(e)$ is the action integral (4.10). From (4.16) it follows that the solution of the problem as escape from the domain G exists of

$$\gamma = \sigma^2 - r > 0, \quad r < \sigma^2 \tag{4.17}$$

and $V_e^0 < 0$. From the first equality of (4.15) and relations (3.7), (4.15) and (4.16) we find

$$V_e^0 = -[2/\gamma I(e)]^{1/2}, \quad u^0 = -rv[2/\gamma I(e)]^{1/2} \tag{4.18}$$

Thus, the control u^0 is equivalent to the introduction of non-linear dissipation into the system.

In the general case of systems of the pendulum-like type, the quantities $I(e)$ and $S(e)$ are expressed in terms of elliptic integrals. The problem is simplified if the purpose of the control is to keep the system far from the potential barrier in a fairly small region hear the equilibrium position. In this case it is possible to linearize the system in the region G: $\{e \leqslant e_r\}$ and assume

$$Q(q) = \frac{(\lambda q)^2}{2}, \quad \omega(e) = \lambda, \quad q = R\cos\Psi, \quad v = -\lambda R \sin\Psi, \quad e = \frac{R^2}{2}$$

Evaluating the action integral (4.10) for the linear system, we obtain $I(e) = \lambda e$. The quasioptimal control (4.18) can then be written in the form

$$u^0 = -2r(\gamma\lambda)^{-1/2} vR^{-1} = 2r(\lambda/\gamma)^{1/2}\sin\Psi \tag{4.19}$$

Thus, the control of the linear system remains non-linear.

It follows from (4.17) that for each value or r there is a corresponding maximum value of the level of perturbations σ for which a solution of the optimal control problem exists at the time interval $\tau_G \sim 1$. If inequality (4.17) is not satisfied, i.e. the level of perturbation is fairly low, the main contribution to the functional (4.13) is made by large deviations with a time of escape $\tau_G \gg 1$. In this case, to solve the problem, we can consider the complete equation (3.4) with the higher-order terms taken into account. In this case, the problem remains singular and functional (4.13) approaches infinity as $\varepsilon \to 0$.

Taking into account that the parameter r determines the level of constraints, it is possible to find the maximum permissible control costs for which the trajectories corresponding to large deviations remain in the admissible region in the time interval $\tau_G \gg 1$.

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